See email for exam 2 stats (well done!).
Closing *Thurs*: 4.4
Closing next *Tues*: 4.4-5
Closing next *Thurs*: 4.7 (last assignment)

Entry Task: **Review**! How would you evaluate these old final questions:

1. $\lim\limits_{x\to 0} \dfrac{e^x - x}{5\cos(x) + 3\sin(x)}$

$= \dfrac{1-0}{5+0} = \boxed{1/5}$

2. $\lim\limits_{x\to 1^+} \dfrac{x - 10}{x(1-x)} = \boxed{+\infty}$

NUM $\to -9$
DEN $\to 0$ THRU NEGATIVE VALUES

$\dfrac{1.000001 - 10}{1.000001(1 - 1.000001)}$ $\dfrac{-}{-}$

3. $\lim\limits_{x\to 4} \dfrac{16 - x^2}{4 - x} \stackrel{0/0}{=} \lim\limits_{x\to 4} \dfrac{(4-x)(4+x)}{(4-x)}$

$= \boxed{8}$

4. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} \stackrel{0/0}{=} \boxed{1}$

5. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x} - 1}{x} \stackrel{0/0}{=}$ CONJUGATE!

$= \lim\limits_{x\to 0} \dfrac{\cancel{1}+x \cancel{-1}}{x(\sqrt{1+x}+1)}$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{1+x}+1} = \boxed{1/2}$

L'Hopital's Rule (0/0 case)

Suppose $g(a) = 0$ and $f(a) = 0$ and *f* and *g* are differentiable at *x* = *a*, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Examples:

1. $\lim\limits_{x\to 4}\dfrac{16-x^2}{4-x} \overset{0/0}{=} \lim\limits_{x\to 4}\dfrac{\cancel{(4-x)}(4+x)}{\cancel{(4-x)}}$

$= \boxed{8}$

Or

$$\lim_{x\to 4}\frac{16-x^2}{4-x} \overset{\boxed{H}}{=} \lim_{x\to 4}\frac{-2x}{-1}$$

$$= \lim_{x\to 4} 2x = \boxed{8}$$

2. $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = \boxed{1}$ ← From 3.3

Or

$$\lim_{x\to 0}\frac{\sin(x)}{x} \overset{0/0}{\underset{\boxed{H}}{=}} \lim_{x\to 0}\frac{\cos(x)}{1} = \boxed{1}$$

3. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x} \overset{0/0}{=} \lim\limits_{x\to 0}\dfrac{\cancel{1}+x\cancel{-1}}{x(\sqrt{1+x}+1)}$

$= \lim\limits_{x\to 0}\dfrac{1}{\sqrt{1+x}+1} = \boxed{\dfrac{1}{2}}$

Or

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} \overset{0/0}{\underset{H}{=}} \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \boxed{\frac{1}{2}}$$

Aside: Sketch of derivation

Assume $g(a) = 0$ and $f(a) = 0$

(These explanations are for the case when g'(a) is not zero).

Explanation 1 (def'n of derivative)

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

provided these limits exist we have:

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Explanation 2 (tangent line approx.):

The tangent lines for $f(x)$ and $g(x)$ at $x = a$ are

$$y = f'(a)(x-a) + 0$$
$$y = g'(a)(x-a) + 0$$

And we know these approximate the functions f(x) and g(x) better and better the closer x gets to a, so

Thus,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

Sometimes you have to use it more than once.

Example:

$$\lim_{x\to 1} \frac{x - \sin(x-1) - 1}{(x-1)^3}$$

$\frac{0}{0}$

$\overset{\boxed{H}}{=} \lim_{x\to 1} \frac{1 - \cos(x-1)}{3(x-1)^2}$

$0/0$

$\overset{\boxed{H}}{=} \lim_{x\to 1} \frac{\sin(x-1)}{6(x-1)}$

$0/0$

$\overset{\boxed{H}}{=} \lim_{x\to 1} \frac{\cos(x-1)}{6}$

$= \boxed{1/6}$

L'Hopitals rule can also be used directly for the ∞/∞ case

2. $\lim_{x\to\infty} \frac{\ln(x)}{x}$ $\frac{\infty}{\infty}$ $= \lim_{x\to\infty} \frac{1/x}{1} = \boxed{0}$

Example:

1. $\lim_{x\to\infty} \frac{(5x+7)}{(6+13x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \frac{5 + 7/x}{6/x + 13} = \boxed{5/13}$

Or

$\lim_{x\to\infty} \frac{5x+7}{6+13x} \overset{\frac{\infty}{\infty}}{\underset{H}{=}} \lim_{x\to\infty} \frac{5}{13} = \boxed{5/13}$

3. $\lim_{x\to\infty} xe^{-3x}$

$= \lim_{x\to\infty} \frac{x}{e^{3x}}$

$\overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{1}{3e^{3x}} = \boxed{0}$

4. $\lim_{x\to\infty} \frac{3x+1}{\sqrt{9+4x^2}}$ $\quad \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \frac{3+\frac{1}{x}}{\sqrt{\frac{9+4x^2}{x^2}}}$ $\qquad \frac{1}{x} = \frac{1}{\sqrt{x^2}}$

$= \lim_{x\to\infty} \frac{3+\frac{1}{x}}{\sqrt{\frac{9}{x^2}+4}} = \frac{3}{\sqrt{4}} = \boxed{3/2}$

Or

$\lim_{x\to\infty} \frac{3x+1}{\sqrt{9+4x^2}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{3}{\left(\frac{8x}{\sqrt{9+4x^2}}\right)}$

$= \lim_{x\to\infty} \frac{3\sqrt{9+4x^2}}{8x}$

oops didn't get easier!

USE THIS METHOD

Other indeterminant forms:

0·∞: (rewrite as a fraction)

$$\lim_{x\to 0^+} x\ln(x) \qquad 0\cdot\infty$$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{\left(\frac{1}{x}\right)} \leftarrow x^{-1}$$

$\frac{-\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to 0^+} -x = \boxed{0}$$

$$\lim_{x\to 0^+} \underset{0}{x}\, \underset{\infty}{e^{1/x}}$$

$$= \lim_{x\to 0^+} \frac{e^{1/x}}{\left(\frac{1}{x}\right)}$$

$\frac{\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{e^{1/x}\cdot\cancel{(-x^{-2})}}{\cancel{(-x^{-2})}}$$

$$= \boxed{\infty}$$

$\infty - \infty$: (combine into a fraction)

$$\lim_{t\to\infty} \frac{2}{t(1+3t)^2} - \frac{2}{t}$$

$$= \lim_{t\to\infty} \frac{2 - 2(1+3t)^2}{t(1+3t)^2}$$

$$= \lim_{t\to\infty} \frac{2 - 2(1+6t+t^2)}{t(1+3t)^2}$$

$$= \lim_{t\to\infty} \frac{-12t - 2t^2}{t(1+3t)^2}$$

$$= \lim_{t\to\infty} \frac{\cancel{t}(-12-2t)}{\cancel{t}(1+3t)^2}$$

$$= \lim_{t\to\infty} \frac{-12-2t}{(1+3t)^2}$$

$$\frac{-\infty}{\infty}$$

$$= \lim_{t\to\infty} \frac{-2}{2(1+3t)\cdot 3} = \boxed{0}$$

$0^0, \infty^0, 1^\infty$: (Use ln( ))

$$\lim_{x\to 0^+} x^x = L = ???$$

$$\ln\left(\lim_{x\to 0^+} x^x\right) = \ln(L)$$

$$\underbrace{\lim_{x\to 0^+} \ln(x^x)} = \ln(L)$$

$$\lim_{x\to 0^+} x\ln(x)$$

$$= \underbrace{\lim_{x\to 0^+} \frac{\ln(x)}{(1/x)}}_{0}$$

$$\Rightarrow \ln(L) = 0$$

$$\Rightarrow \boxed{L = e^0 = 1}$$

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = L$$

$$\underbrace{\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)} = \ln(L)$$

$$\lim_{x\to\infty} \frac{\left(\ln\left(1+\frac{2}{x}\right)\right)}{\left(1/x\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+2/x}\left(\cancel{-}\frac{2}{\cancel{x^2}}\right)}{\cancel{\left(-1/x^2\right)}}$$

$$= \frac{2}{1+0} = 2$$

$$\Rightarrow \quad 2 = \ln(L)$$

$$\boxed{L = e^2}$$

Aside (you don't need to know this): This is an important application of what we just discussed:

The formula for compound interest is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P = starting balance,

A = end balance,

r = annual rate,

n = number of times interest is compounded each year,

t = number of years

In some bank accounts interest is computed once a month, for some every day, for some every second. If you wanted interested to always be computed (continuously), then the new formula would be

$$A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt}$$

You can use the techniques just discussed to find this limit and you get $\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = Pe^{rt}$

Thus,

$$A = Pe^{rt}$$

is the *continuous compounding* interest formula.